Bi-connexive variants of Heyting-Brouwer logic

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In these lectures, I intend to present material from two still unpublished papers. The first lecture will be based on [12], and the second talk will be based on [49].

1 First talk

Systems of *connexive logic* and the *bi-intuitionistic* logic BiInt that is also known as *Heyting-Brouwer logic* have been carefully studied since the 1960s and 1970s with various philosophical and mathematical motivations, see [2, 19, 20, 47] and [29, 30, 31, 13, 8]. The characteristic principles of connexive logic are usually traced back to Aristotle and Boethius, and the co-implication of BiInt can be traced back to Skolem [35].

A distinctive feature of connexive logics is that they validate the so-called

Aristotle's theses: $\sim(\alpha \to \sim\alpha)$ and $\sim(\sim\alpha \to \alpha)$, and
Boethius' theses: $(\alpha \to \beta) \to \sim(\alpha \to \sim\beta)$ and $(\alpha \to \sim\beta) \to \sim(\alpha \to \beta)$.

An intuitionistic (or constructive) connexive modal logic, CK, which is a constructive connexive analogue of the smallest normal modal logic K, was introduced in [43] by extending a certain basic intuitionistic (or constructive) connexive logic, C, which is a connexive variant of *Nelson's paraconsistent logic* [1, 21, 23, 11].[1] A *classical connexive modal logic* called CS4, which is based on the positive normal modal logic S4, was introduced in [9] as a Gentzen-type sequent calculus. The Kripke-completeness and cut-elimination theorems for CS4 were shown, and CS4 was shown to be embeddable into positive S4 and to be decidable. Moreover, it was shown in [9] that the basic constructive connexive logic C can be faithfully embedded into CS4 and into a subsystem of CS4 lacking syntactic duality between necessity and possibility.

Heyting-Brouwer logic, which is an extension of both *dual-intuitionistic logic*, DualInt, and intuitionistic logic, Int, was introduced by Rauszer [29, 30, 31], who proved algebraic and Kripke completeness theorems for BiInt. As was shown by Uustalu in 2003, cf.[27], the original Gentzen-type sequent calculus by Rauszer

[1] Information on connexive logics can also be found on the web site [26].

[29] does not enjoy cut-elimination, and various kinds of sequent systems for BiInt have been presented in the literature, including cut-free display sequent calculi in [8, 44], see also [28] and [27] for a comparison between sequent calculi for BiInt. Moreover, BiInt is known to be a logic that has a faithful embedding into the future-past tense logic KtT4 [14], and a modal logic based on BiInt was studied by Łukowski in [15].

Dual-intuitionistic logics are logics which have a Gentzen-type sequent calculus in which sequents have the restriction that the antecedent contains at most one formula [3, 7, 36]. This restriction of being singular in the antecedent is syntactically dual to that in Gentzen's sequent calculus LJ for intuitionistic logic, which is singular in the consequent. Historically speaking, the logics in the set of logics containing Czermak's *dual-intuitionistic calculus* [3], Goodman's *logic of contradiction* or *anti-intuitionistic logic* [7], and Urbas's extensions of Czermak's and Goodman's logics [36] were collectively referred to by Urbas as dual-intuitionistic logics. The dual-intuitionistic logic referred to as DualInt in [8, 46] is the implication-free fragment of BiInt. An interpretation of DualInt as the *logic of scientific research* was presented by Shramko in [32].

In this talk we combine the two approaches and introduce the *bi-intuitionistic connexive logic* (or *connexive Heyting-Brouwer logic*), BCL, as a Gentzen-type sequent calculus. The logic BCL may be seen as an extension of the connexive logic C from [43] by the co-implication of BiInt, using a connexive understanding of negated co-implications. Another understanding of co-implication is developed in [46, 48], and a natural deduction proof system and formulas-as-types notion of construction for a bi-connexive logic 2C that assumes this understanding of co-implication is presented in [49] and in the second talk.

We will proceed as follows. In a first step, the logic BCL is introduced as a Gentzen-type sequent calculus, and a dual-valuation-style Kripke semantics for BCL is defined. BCL is constructed on the basis of Takeuti's cut-free Gentzen-type sequent calculus LJ′ for Int. Gentzen-type sequent calculi ICL, DCL, BL, IL and DL for *intuitionistic connexive logic*, *dual-intuitionistic connexive logic*, *bi-intuitionistic logic*, *intuitionistic logic* and *dual-intuitionistic logic*, respectively, are defined as subsystems of BCL.

In a second step, some theorems for syntactically and semantically embedding BCL into BL are proved, and using these theorems, the completeness theorem with respect to the Kripke semantics for BCL is shown as a central result. The cut-elimination theorems for ICL and DCL are shown using some restricted versions of the syntactical embedding theorem of BCL into BL. The cut-elimination theorem does *not* hold for BCL and BL.

Next, some theorems for syntactically embedding ICL into DCL and vice versa are shown. These theorems reveal that ICL and DCL are syntactically dual to each other in a certain sense. Thus, it is shown in these theorems that BCL is constructed based on a duality principle of the characteristic subsystems.

Finally, we present a sound and complete tableau calculus for BCL and its subsystems ICL, DCL, BL, IL, and DL using triply-signed formulas.

2 Second talk

In various branches of non-classical logic a distinction is drawn between truth and falsity as concepts that are primitive and independent of each other, though not necessarily disconnected. The separation of truth and falsity is achieved by giving up bivalence, so that falsity is distinguished from the absence of truth and truth is discriminated from the absence of falsity. In many-valued logic this leads to a distinction between a set of designated values and a set of antidesignated values and, moreover, to a multiplicity of entailment relations. In addition to the familiar preservation of designated values, there is, for example, q-entailment, "quasi-entailment", that leads from not-antidesignated premises to a designated conclusion (see [16, 17, 18]) and p-entailment, "pausibility-entailment", that leads from designated premises to a not-antidesignated conclusion (see [5, 6]). Quasi-entailment and plausibility-entailment are peculiar insofar as entailment is not defined in terms of preservation of membership in some subset of the set of truth values (alias truth degrees). In [34, p. 210], preservation of not being antidesignated from the premises to the conclusion of an inference is listed as an intuitively appealing notion of entailment. If truth (being designated) and falsity (being antidesignated) are treated on a par and not as each other's complement, then it makes much sense to take falsity-preservation from the premises to the conclusion of an inference very seriously as well.

In the logic of generalized truth values (see [33, 25], [34]), entailment relations are defined with respect to a partial order on a set of semantical values, in particular as relations defined on a set generated from the set **2** of classical truth values by iterated powerset formation. This approach very naturally leads to a distinction between a truth ordering and a separate falsity ordering on the powerset of the powerset of **2** together with two distinct entailment relations, truth entailment and falsity entailment.[2]

Whilst usually a notion of falsity is internalized into the logical object language by means of a negation connective, the notion of truth typically is *not* internalized at all. This is only one out of many ways in which "positive" concepts traditionally predominate even in non-classical logic in comparison to their "negative" counterparts. It would be possible to consider a formula A not as a vehicle for making an assertion, but rather as a device for making a denial. From the latter perspective one would be interested in having available a unary connective that internalizes truth instead of falsehood. The internalization of truth and falsity can be realized by a division of labour, namely by using two different negation connectives, or by utilizing a single one that internalizes both truth from the point of view of falsification and falsity from the perspective of verification. The former is achieved in the bi-intuitionistic logic **2Int** from [46]. In that system two negation operations are defined: intuitionistic negation and a connective that is in a certain sense dual to intuitionistic negation. Intuitionistic negation is negation as "implies falsity"; its dual, called co-negation, is understood as "co-implies truth". The intuitionistic negation

[2] Moreover, the subset relation on the set of generalized truth values may be seen as an information ordering.

$\neg A$ of a formula A internalizes an indirect notion of falsification into the logical object language from the point of view of verification: a state supports the truth of $\neg A$ iff A implies falsity (iff the assumption that A is true leads to the truth of the falsity constant $\bot$). The co-negation $-A$ of a formula A internalizes an indirect notion of verification into the object language from the point of view of falsification: a state supports the falsity of $-A$ iff A co-implies truth (iff the assumption that A is false leads to the falsity of the truth constant $\top$). The two negations are *defined* using the zero-place connectives $\top$ and $\bot$. One may, however, use a *primitive* strong negation, $\sim$, as in Nelson's constructive logics with strong negation **N3**, **N4** , and $\mathbf{N4}^{\bot}$ (see, for example, [21, 22, 1, 38, 39, 4, 40, 42, 24, 23, 10, 11]) that provides both internalizations and thereby turns dual provability into a relation of disprovability, cf. [45].

Nevertheless, there is a certain preoccupation with the positive dimension of logic even in Nelson's systems and a lacuna in the separate treatment of truth and falsity. The constructive implication in Nelson's logics internalizes an entailment relation that preserves support of truth from the premises to the conclusion of an inference or, proof-theoretically, internalizes a corresponding derivability relation. However, in Nelson's logics there is no connective that internalizes the preservation of support of falsity from the premises to the conclusion of an inference or, proof-theoretically, internalizes a corresponding relation of dual derivability. Such a dual of implication, called co-implication, is present in the system **2Int** from [46, 48].

In this talk, the bi-intuitionistic system **2Int** is modified: a primitive strong negation is added that internalizes falsity with respect to verification and truth with respect to falsification. Moreover, for this strong negation a connexive reading of negated implications and co-implication is assumed, so that **2C** also emerges as an extension of the connexive propositional logic C from [41], which was obtained from propositional **N4** by replacing the familiar falsification condition for negated implications by its connexive version.

The reason for considering *connexive* implication, $\rightarrow$, and *connexive* co-implication, $\prec$, instead of assuming the familiar understanding of negated implications in Nelson's and other logics is that one obtains a neat encoding of derivations in the $\{\rightarrow, \prec, \sim\}$-fragment of the language under consideration by typed λ-terms built up from atomic terms of two sorts, one for proofs and one for dual proofs, using only (i) functional application, (ii) functional abstraction, and (iii) certain sort/type-shift operations that turn an encoding of a dual proof of a formula A [respectively $\sim A$] into an encoding of a proof of $\sim A$ [respectively A] and that turn an encoding of a proof of a formula A [respectively $\sim A$] into an encoding of a dual proof of $\sim A$ [respectively A]. In [37, 39] an encoding of derivations in Nelsons's constructive logics with strong negation **N3** and **N4** was obtained by giving up the unique typedness of terms. The use of terms of two sorts avoids this feature: every term is uniquely typed.

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