First-Order Logic with Imperfect Information

Allen L. Mann

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Game-theoretic semantics defines truth and satisfaction in terms of (semantic) games. Although the fundamental intuition that quantifiers can be interpreted as moves in a game appears as early as Peirce's second Cambridge Conferences lecture [8] and Henkin's seminal paper on branching quantifiers [2], game-theoretic semantics was first popularized by Hintikka [3, 4] (see also [5, 7]).

The semantic game associated with a first-order sentence is a contest between two opponents. One player tries to verify the sentence by choosing the values of existentially quantified variables, while the other tries to falsify it by picking the values of universally quantified variables. Disjunctions prompt the existential player to choose which disjunct to verify; conjunctions prompt the universal player to pick which conjunct to falsify. Negation tells the players to switch roles. A first-order sentence is true (false) in a suitable structure if and only if the existential (universal) player has a winning strategy.

In order to define the semantic game for an open first-order formula, one must specify the values of its free variables. Usually, this is done using an assignment. If the open formula in question is a subformula of some first-order sentence, then we can think of the assignment as encoding the previous moves of the players in the semantic game for the sentence.

In the semantic game for a first-order formula, the players take turns making their moves, and at each decision point the active player is aware of every move leading up to the current position. Thus semantic games can be modeled as extensive games with perfect information.

First-order logic with imperfect information is an extension of first-order logic obtained by considering semantic games with imperfect information. In a game with imperfect information, the active player may not be aware of every move leading up to the current position. To specify such games, we must extend the syntax of first-order logic to be able to indicate what information is available to the active player. We briefly describe two approaches found in the literature.

Independence-friendly (IF) logic, introduced by Hintikka and Sandu [6], adds a slash set to each quantifier that indicates which variables the active player is not allowed to access when choosing the value of the quantified variable. For example, in the independence-friendly sentence

$$\forall x \big(\exists y/\{x\}\big) Rxy,$$

the existential player must choose the value of y without knowing the value of x.

Dependence logic, introduced by Väänänen [9], utilizes new atomic formulas of the form

$$=\!(t_1, \ldots, t_n)$$

whose intuitive meaning is that the value of the term t_n is determined by the values of the terms $t_1, \ldots, t_{n-1}$. The atomic formula $=\!(t)$ asserts that the value of t is constant. Thus, when playing the semantic game for the dependence logic formula

$$\forall x \exists y \big(=\!(y) \land Rxy\big),$$

the existential player knows the value of x when choosing the value of y, but if the game is repeated she must choose the same value for y as before, regardless of the new value of x.

Both IF logic and dependence logic have the same expressive power as existential second-order logic, a result first proved independently by Enderton [1] and Walkoe [10] in the context of first-order logic with branching quantifiers.

References

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