Game Theory

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Extensive games

Definition

An *extensive game (with perfect information)* has the following components:

- N, a set of *players*
- H, a set of *histories* or *plays*
- Z, a set of *terminal histories* or *complete plays*
- P, a function that assigns a player to each nonterminal history
- u_p, a *utility function* for each player

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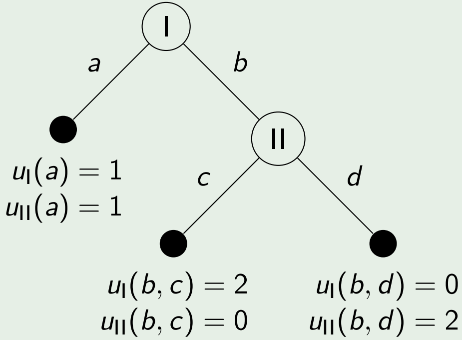
Extensive games

Example

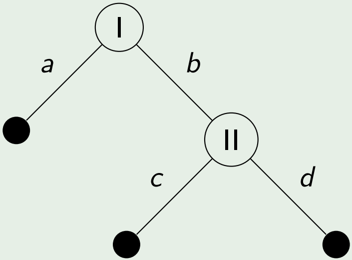
An extensive game (with perfect information):



Extensive games

Example

An extensive game form (with perfect information):



Strategies

Definition

- H_p is the set of histories where it is player p's move.
- If $h = (a_1, \ldots, a_n)$, then
$$A(h) = \{\, a : (a_1, \ldots, a_n, a) \in H \,\}.$$
- A *strategy for player p* is a choice function
$$\sigma \in \prod_{h \in H_p} A(h)$$
that tells the player how to move whenever it is his or her turn.

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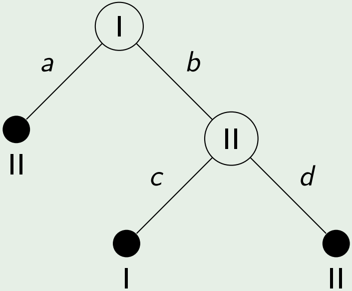
Win-lose extensive games

Example

A win-lose extensive game:



Determined games

Definition

- A strategy is *winning* if its owner wins every terminal history in which he or she follows it.
- A win-lose game is *determined* if one of the players has a winning strategy.

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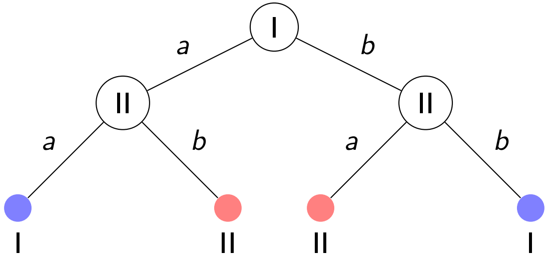
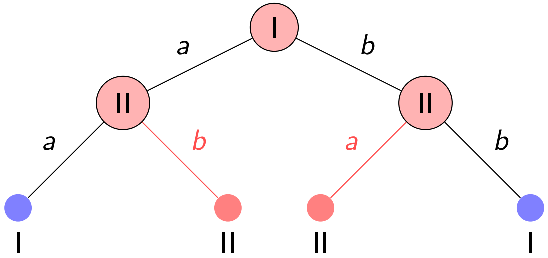
Determined games

Is this game determined?



Determined games

Is this game determined? YES



Determined games

Theorem (Gale-Stewart)

Every closed game is determined.

Corollary

Every two-player, win-lose, extensive game with perfect information that has finite horizon (and a unique initial history) is determined.

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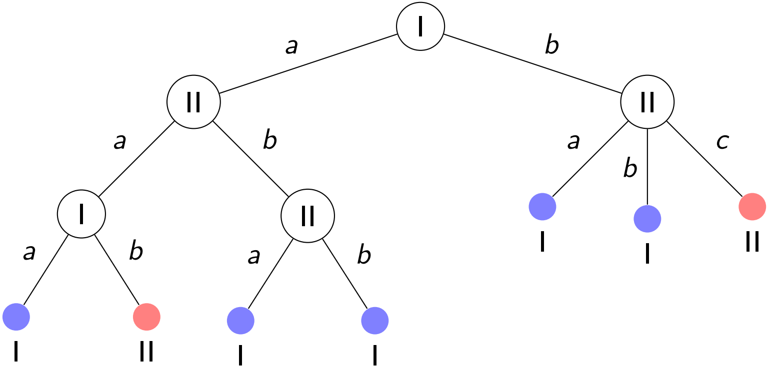
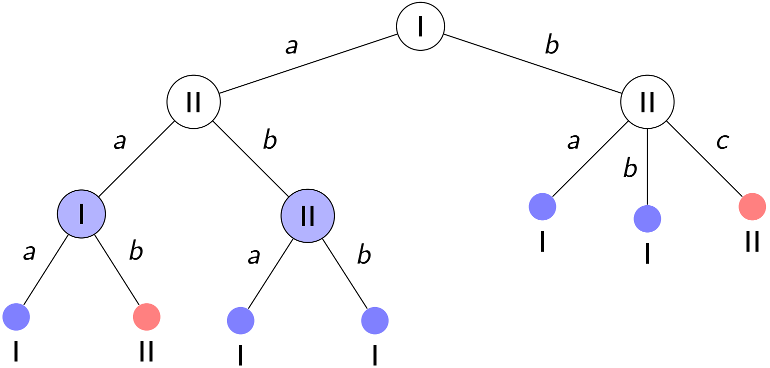
Gale-Stewart Theorem



Gale-Stewart Theorem



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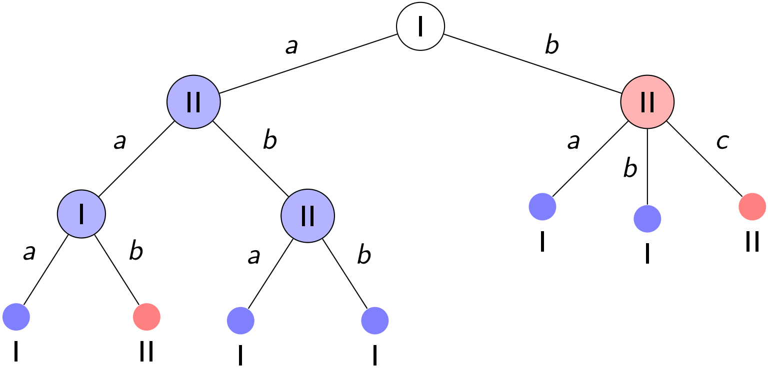
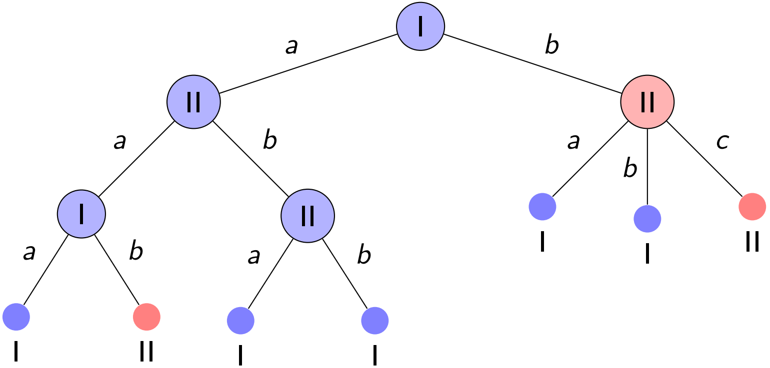
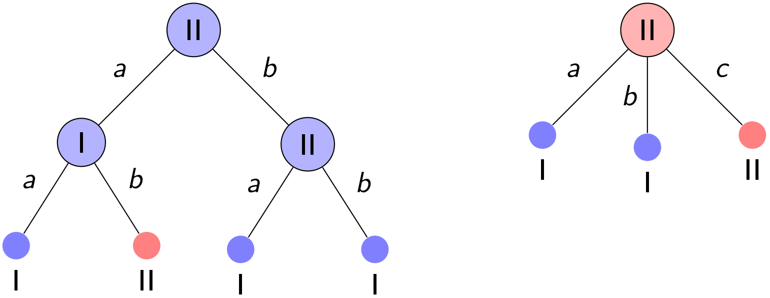


Gale-Stewart Theorem



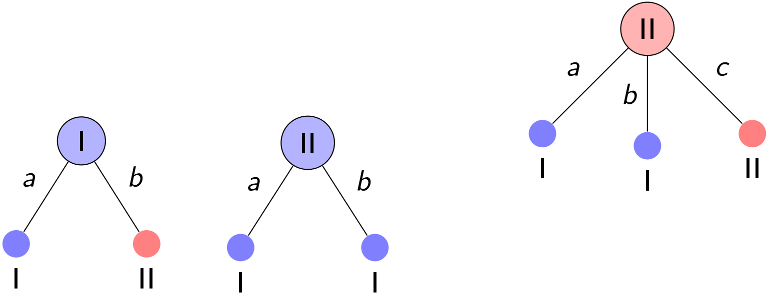
Subgames

Player I has a winning strategy for the original game if and only if she has a winning strategy for at least one of these two subgames.



Subgames

Player I has a winning strategy for the original game iff she wins both sub-subgames on the left *or* the subgame on the right.



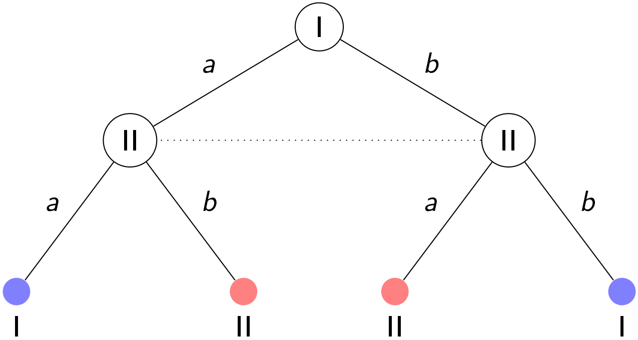
Extensive games (with imperfect information)

Definition

An *extensive game (with imperfect information)* has an additional indistinguishability relation $\sim_p$ for each player.

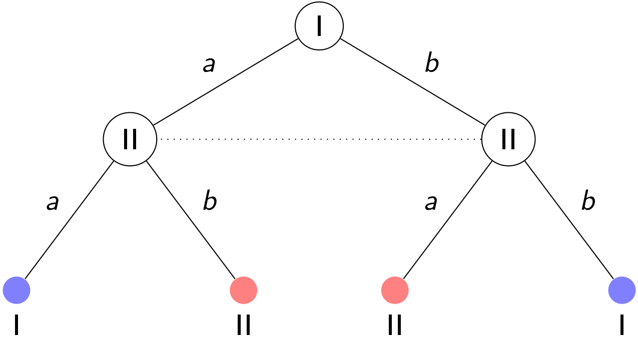
Extensive games (with imperfect information)

An extensive game (with imperfect information):



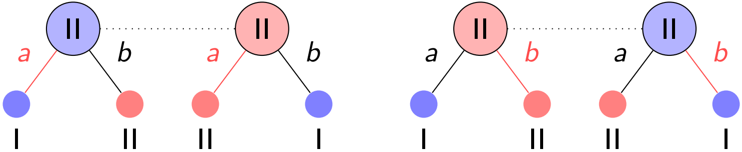
Extensive games (with imperfect information)

Is it determined?



Extensive games (with imperfect information)

Is it determined? NO



Strategic games

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A *strategic game* has the following components:

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Stag Hunt

Example

Player III = stag

	stag	hare
stag	2, 2, 2	0, 1, 0
hare	1, 0, 0	1, 1, 0

Player III = hare

	stag	hare
stag	0, 0, 1	0, 1, 1
hare	1, 0, 1	1, 1, 1

Prisoners' Dilemma

	quiet	fink
quiet	$-1, -1$	$-4, 0$
fink	$0, -4$	$-3, -3$

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Strictly competitive

Definition

A two-player game is *strictly competitive* if for all $\sigma, \sigma' \in S_{\mathrm{I}}$ and $\tau, \tau' \in S_{\mathrm{II}}$ we have

$$u_{\mathrm{I}}(\sigma, \tau) \leq u_{\mathrm{I}}(\sigma', \tau') \quad \text{iff} \quad u_{\mathrm{II}}(\sigma, \tau) \geq u_{\mathrm{II}}(\sigma', \tau').$$

Matching Pennies

	heads	tails
heads	$1, \ -1$	$-1, \ \ 1$
tails	$-1, \ \ 1$	$1, \ -1$

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heads	$1, -1$	$-1, 1$
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Mixed strategies

Definition

- A *mixed strategy* for player p is a probability distribution over S_p.
- The *expected utility function* for player p is

$$U_p(\mu, \nu) = \sum_{\sigma \in S_{\mathrm{I}}} \sum_{\tau \in S_{\mathrm{II}}} \mu(\sigma)\, \nu(\tau)\, u_p(\sigma, \tau).$$

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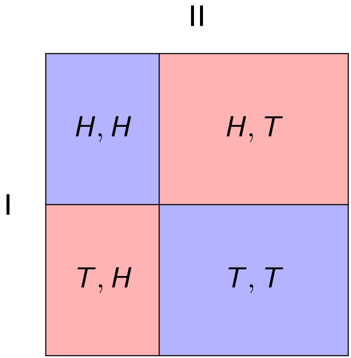
Mixed equilibrium



Mixed equilibrium

II

I	H, H	H, T
	T, H	T, T

Mixed equilibrium

Definition

A pair (μ^*, ν^*) of mixed strategies is an *equilibrium* if:

- $U_{\mathrm{I}}(\mu, \nu^*) \leq U_{\mathrm{I}}(\mu^*, \nu^*)$ for every mixed strategy $\mu \in \Delta(S_{\mathrm{I}})$,
- $U_{\mathrm{II}}(\mu^*, \nu) \leq U_{\mathrm{II}}(\mu^*, \nu^*)$ for every mixed strategy $\nu \in \Delta(S_{\mathrm{II}})$.

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Multiple equilibria

Theorem

If (μ, ν) and (μ', ν') are two mixed equilibria for a strictly competitive game, then $U_p(\mu, \nu) = U_p(\mu', \nu')$.

Proof.

If (μ, ν) and (μ', ν') are both equilibria, then

$$U_{\mathrm{I}}(\mu, \nu) \leq U_{\mathrm{I}}(\mu, \nu') \leq U_{\mathrm{I}}(\mu', \nu') \leq U_{\mathrm{I}}(\mu', \nu) \leq U_{\mathrm{I}}(\mu, \nu).$$

Hence $U_{\mathrm{I}}(\mu, \nu) = U_{\mathrm{I}}(\mu', \nu')$. Similarly, $U_{\mathrm{II}}(\mu, \nu) = U_{\mathrm{II}}(\mu', \nu')$. $\square$

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Minimax theorem

Theorem (Von Neumann)

Every finite, two-player, zero-sum game has an equilibrium in mixed strategies.

Definition

The *value* of such a game is $U_I(\mu^*, \nu^*)$, where (μ^*, ν^*) is any equilibrium.

Identifying equilibria

Theorem

In a finite two-player game, a pair of mixed strategies (μ^*, ν^*) *is an equilibrium if:*

- $U_{\mathrm{I}}(\mu^*, \nu^*) = U_{\mathrm{I}}(\sigma, \nu^*)$ *for every* σ *in the support of* μ^**,*
- $U_{\mathrm{I}}(\mu^*, \nu^*) \geq U_{\mathrm{I}}(\sigma, \nu^*)$ *for every* σ *not in the support of* μ^**,*
- $U_{\mathrm{II}}(\mu^*, \nu^*) = U_{\mathrm{II}}(\mu^*, \tau)$ *for every* τ *in the support of* ν^**,*
- $U_{\mathrm{II}}(\mu^*, \nu^*) \geq U_{\mathrm{II}}(\mu^*, \tau)$ *for every* τ *outside the support of* ν^**.*

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