First-order logic with imperfect information

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Applications of Logic in Philosophy and Foundations of Mathematics

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Syntax

Definition

Given a fixed vocabulary and a set of variables, a *first-order language* is generated by the following grammar:

$$\alpha \mid \neg\varphi \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \exists x \varphi \mid \forall x \varphi$$

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Given a fixed vocabulary and a set of variables, a *first-order language* is generated by the following grammar:

$$\alpha \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\varphi \mid \forall x\varphi$$

Game-theoretic semantics

Example

$$\forall x \exists y (x = y)$$

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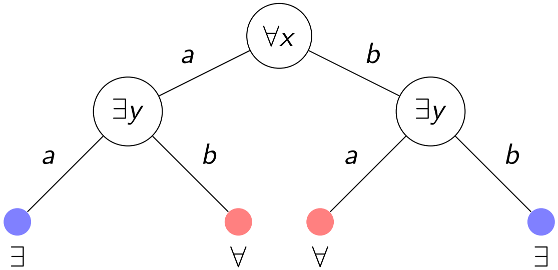

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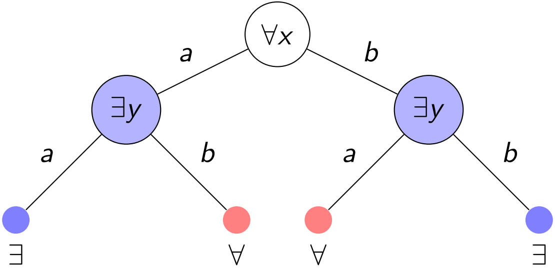

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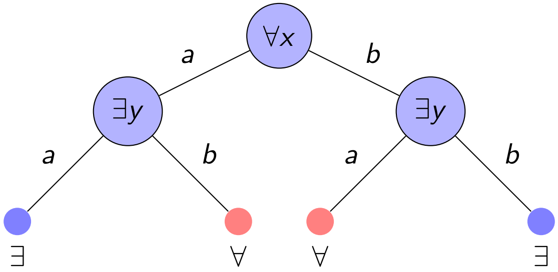
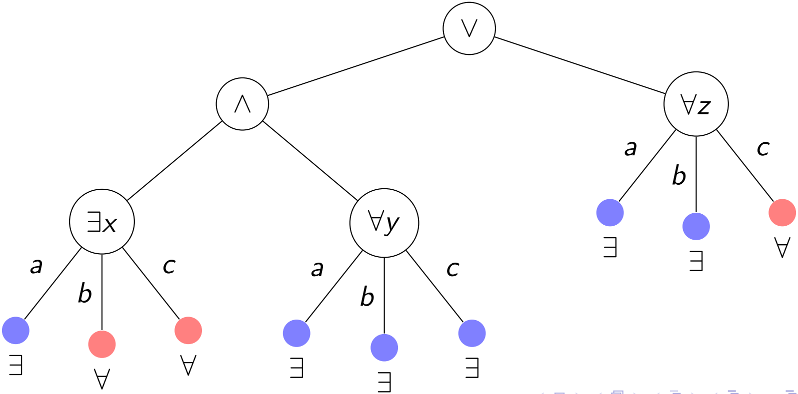


Game-theoretic semantics

Example

$$(\exists x Px \land \forall y Qy) \lor \forall z Rz$$



Game-theoretic semantics

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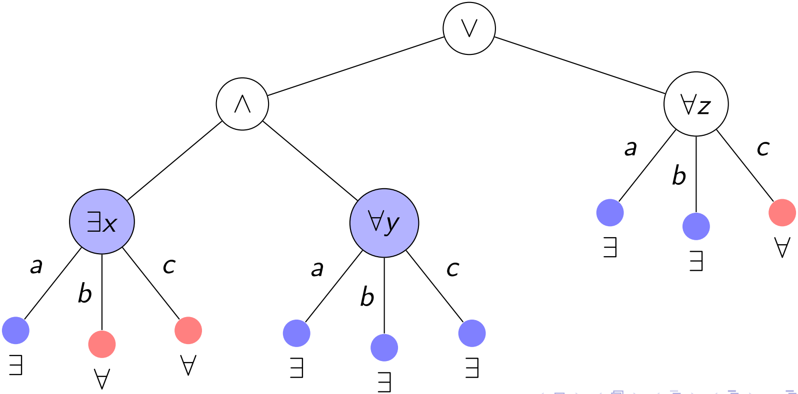

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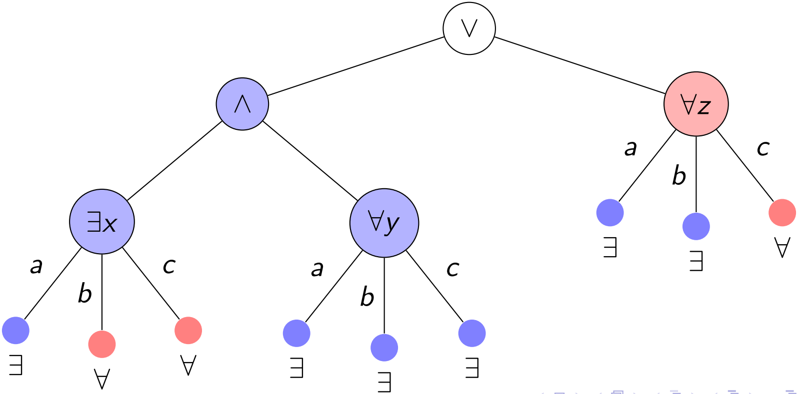

Game-theoretic semantics

Example

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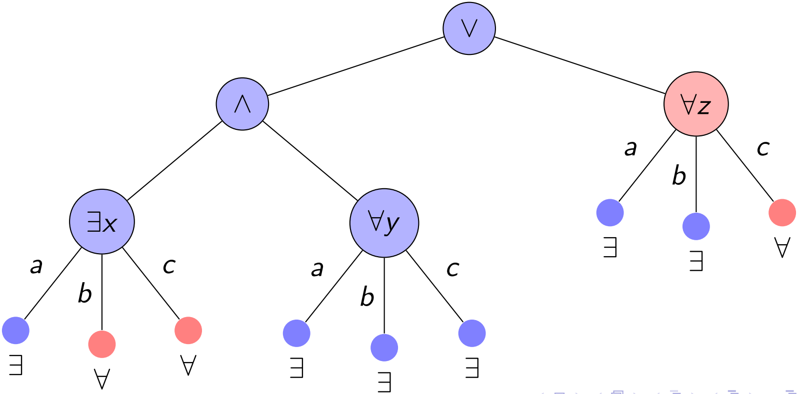
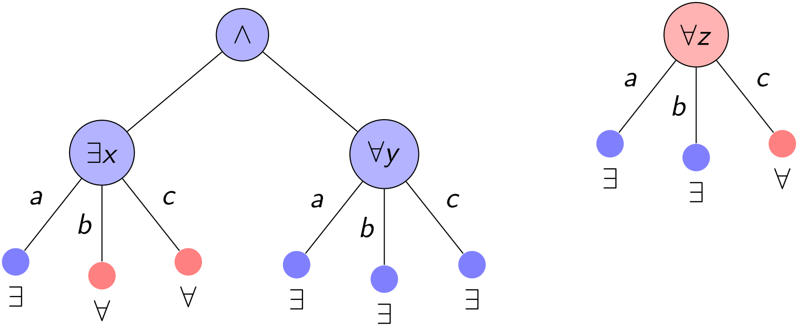


Tarski's semantics

Example

$$(\exists x Px \wedge \forall y Qy) \quad \text{or} \quad \forall z Rz$$



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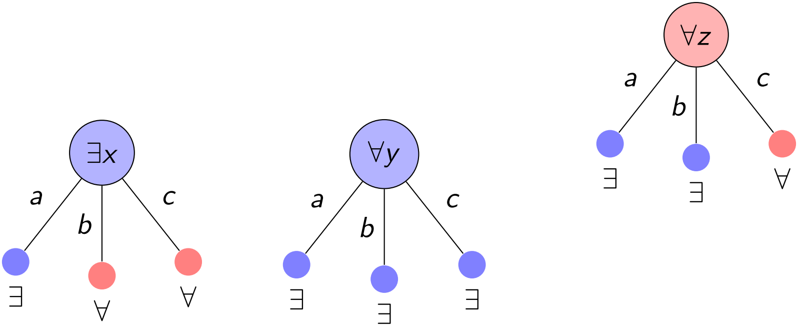

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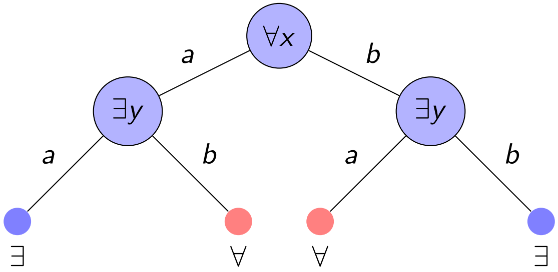
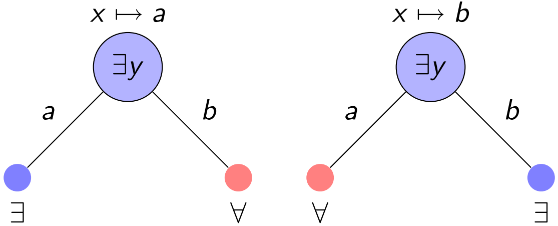
$$\forall x \exists y (x = y)$$



Tarski's semantics

Example

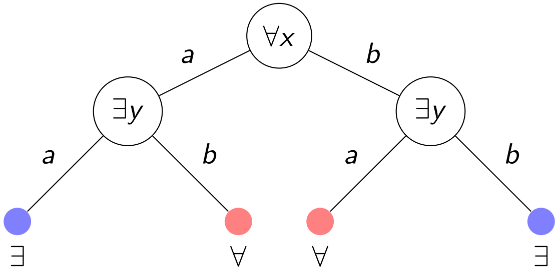
$$\text{for all } x,\ \exists y(x = y)$$



Skolem functions

Example

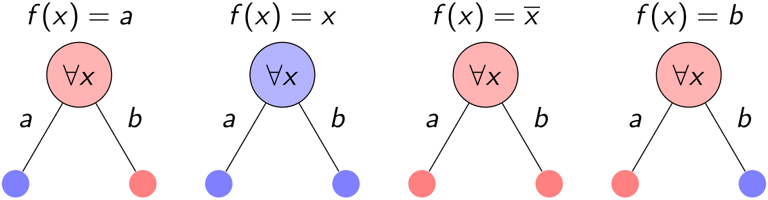
$$\forall x \exists y (x = y)$$



Skolem functions

Example

$$\forall x(x = f(x))$$



Skolemization (outside-in)

Normally we Skolemize a sentence from outside in...

Example

$$\forall x \exists y \big(x < y \lor \exists z (y < z)\big)$$

$$\forall x \big[x < f(x) \lor f(x) < g(x)\big]$$

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Skolemization (inside-out)

But we can also Skolemize from inside-out.

Definition

Let U be a finite set of variables containing $\mathsf{Free}(\varphi)$.

$$\begin{array}{rcll}
\mathsf{Sk}_U(\psi) & \text{is} & \psi & (\psi \text{ atomic}), \\
\mathsf{Sk}_U(\neg\psi) & \text{is} & \neg\,\mathsf{Sk}_U(\psi) & (\psi \text{ atomic}), \\
\mathsf{Sk}_U(\psi \circ \psi') & \text{is} & \mathsf{Sk}_U(\psi) \circ \mathsf{Sk}_U(\psi'), & \\
\mathsf{Sk}_U(\exists x \psi) & \text{is} & \mathsf{Subst}\big(\mathsf{Sk}_{U \cup \{x\}}(\psi), x, f_{\exists x \psi}(y_1, \ldots, y_n)\big), & \\
\mathsf{Sk}_U(\forall x \psi) & \text{is} & \forall x\, \mathsf{Sk}_{U \cup \{x\}}(\psi), &
\end{array}$$

where $y_1, \ldots, y_n$ enumerates the variables in U.

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Example

$$\forall x \exists y \big(x < y \vee \exists z(y < z)\big)$$

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\mathrm{Sk}_{\{x,y,z\}}(y < z) \quad &\text{is} \quad y < z, \\
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\mathrm{Sk}_{\{x\}}\Big[\exists y\big(x < y \vee \exists z(y < z)\big)\Big] \quad &\text{is} \quad x < f(x) \vee f(x) < g\big(x, f(x)\big).
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Syntax

Definition

Given a fixed vocabulary and a set of variables, an *independence-friendly (IF) language* is generated by the following grammar:

$$\alpha \mid \neg\alpha \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid (\exists x/W)\varphi \mid (\forall x/W)\varphi$$

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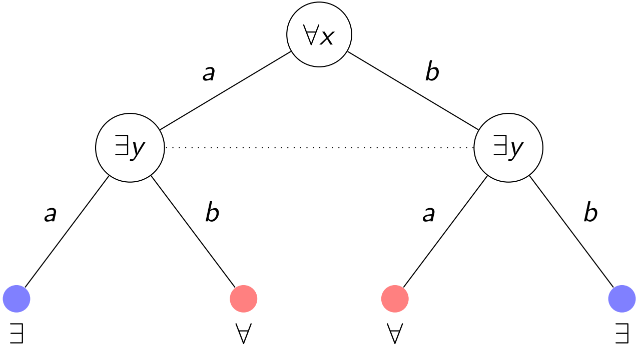
Game-theoretic semantics

Example

$$\forall x (\exists y/\{x\})\, x = y$$



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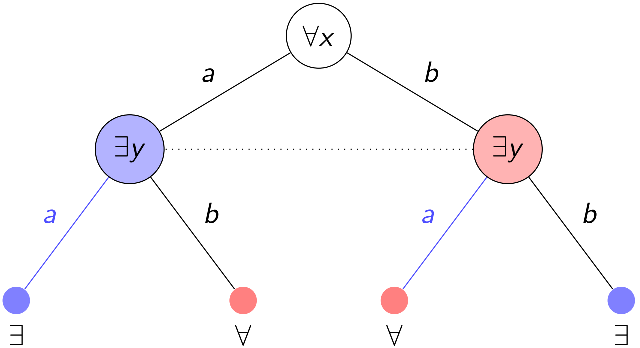

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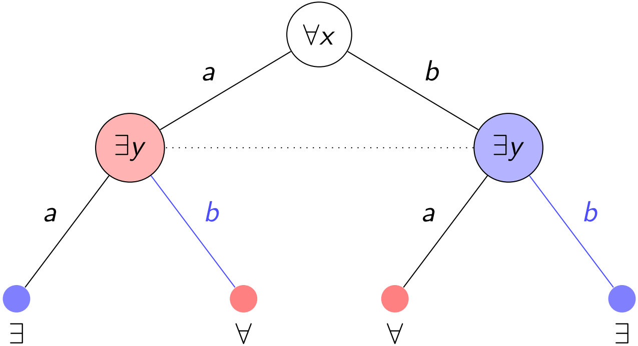

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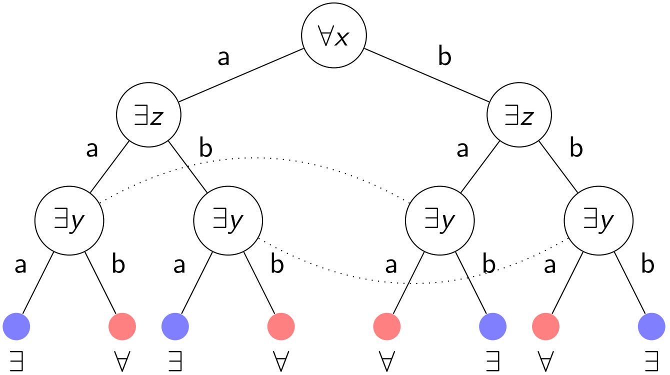

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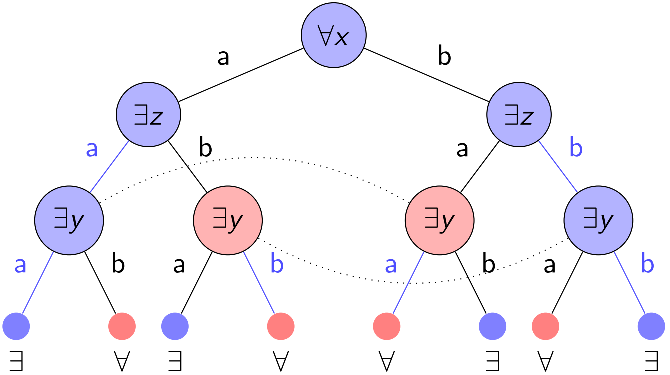

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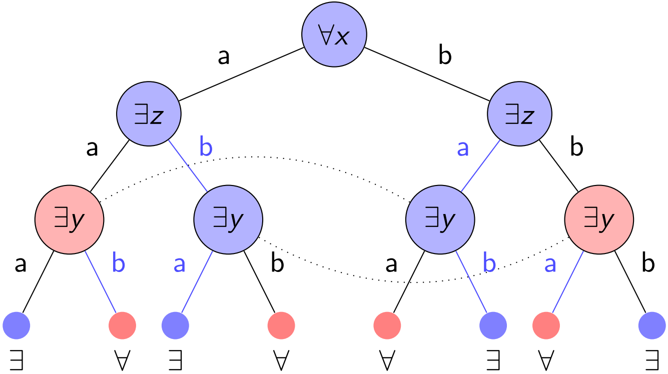

Trump semantics

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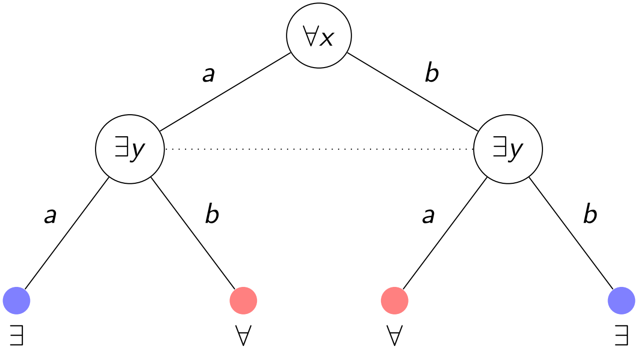

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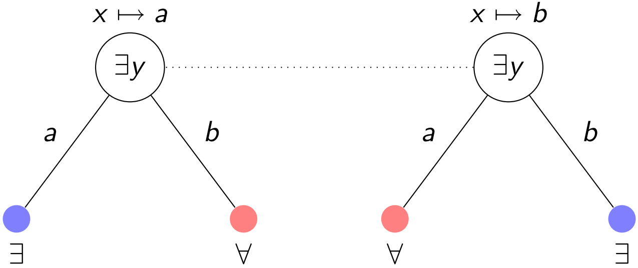

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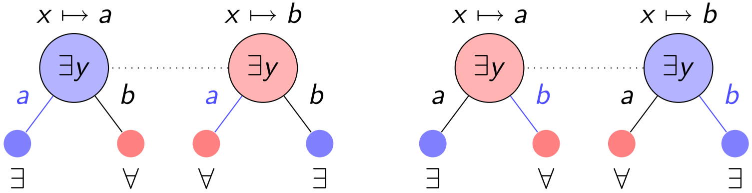

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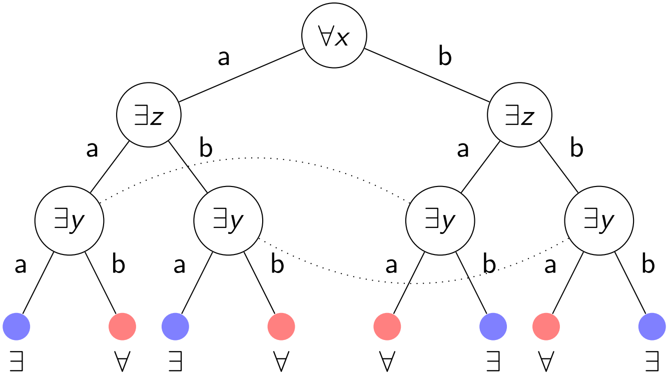

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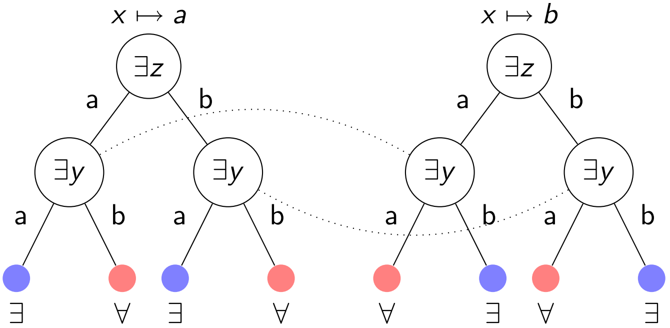

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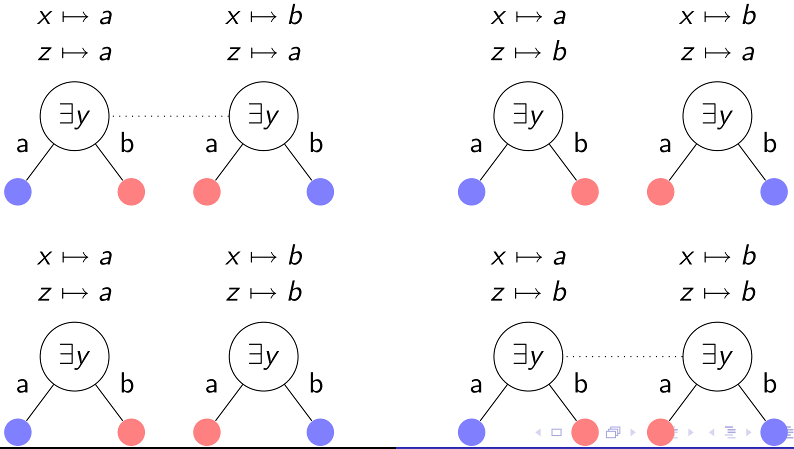
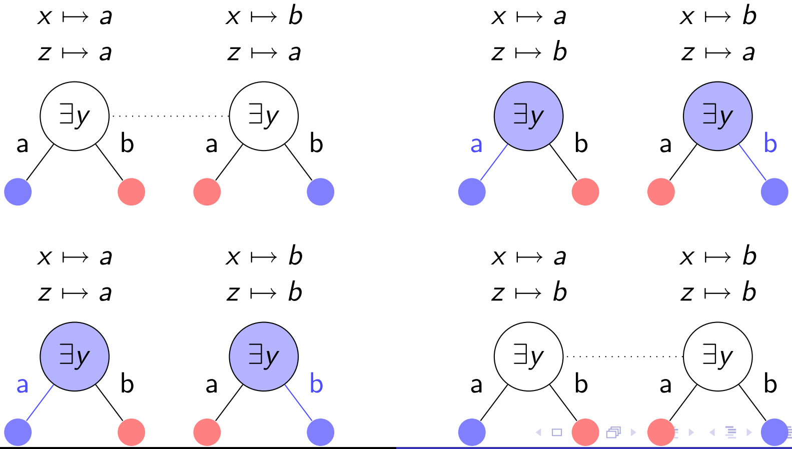


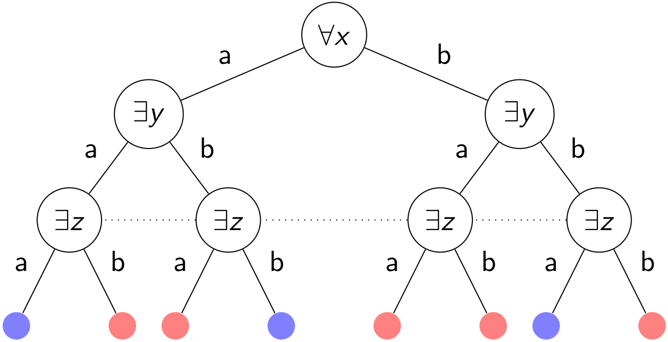
Trump semantics

$$\left(\exists y/\{x\}\right) x = y$$



Trump semantics

$$\forall x \exists y \left(\exists z / \{x, y\} \right) Rxyz$$



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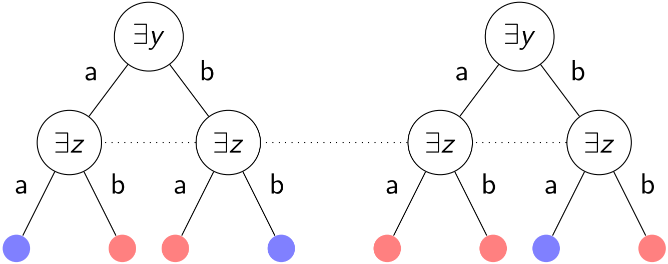

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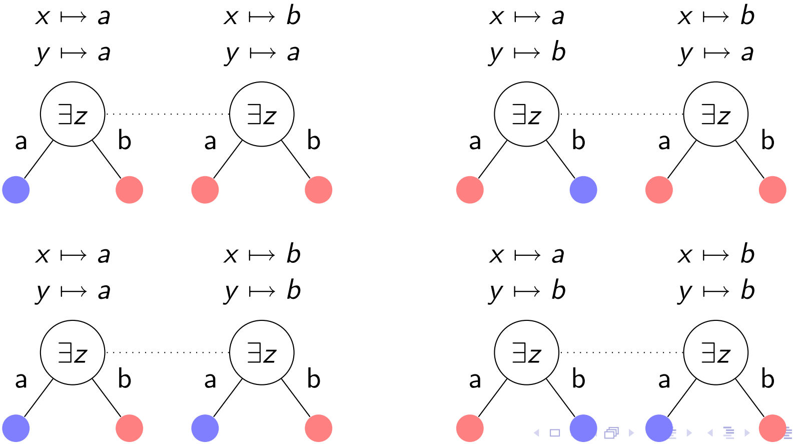

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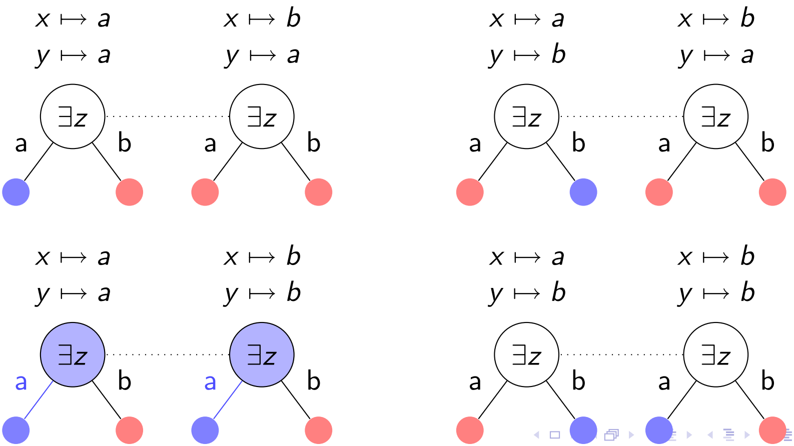

Skolem semantics

Definition

Let U be a finite set of variables containing $\mathrm{Free}(\varphi)$.

$$
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\mathrm{Sk}_U(\psi) & \text{is} & \psi \quad (\psi \text{ literal}), \\
\mathrm{Sk}_U(\psi \circ \psi') & \text{is} & \mathrm{Sk}_U(\psi) \circ \mathrm{Sk}_U(\psi'), \\
\mathrm{Sk}_U\big((\exists x/W)\psi\big) & \text{is} & \mathrm{Subst}\big(\mathrm{Sk}_{U\cup\{x\}}(\psi), x, f_{(\exists x/W)\psi}(y_1, \dots, y_n)\big), \\
\mathrm{Sk}_U\big((\forall x/W)\psi\big) & \text{is} & \forall x\, \mathrm{Sk}_{U\cup\{x\}}(\psi),
\end{array}
$$

where $y_1, \dots, y_n$ enumerates the variables in $U - W$.

Skolem semantics

Theorem

$$\mathbf{M} \models^{+} \varphi \quad \textit{iff} \quad \mathbf{M}^{*} \models \mathrm{Sk}(\varphi)$$

for some expansion $\mathbf{M}^{*}$ *of* $\mathbf{M}$ *to the vocabulary*

$$L^{*} = L \cup \{\, f_{\psi} : \psi \in \mathrm{Subf}_{\exists}(\varphi) \,\}.$$

Skolem semantics

Theorem

$$\mathbf{M} \models^{-} \varphi \quad \textit{iff} \quad \mathbf{M}^{*} \models \mathrm{Sk}(\neg\varphi)$$

for some expansion $\mathbf{M}^{*}$ *of* $\mathbf{M}$ *to the vocabulary*

$$L^{*} = L \cup \{\, f_{\psi} : \psi \in \mathrm{Subf}_{\forall}(\varphi) \,\}.$$

Skolem semantics

Example

$$\forall x \big(\exists y/\{x\}\big) x = y$$

$$\begin{aligned}
\mathrm{Sk}_{\{x,y\}}(x = y) \quad &\text{is} \quad x = y, \\
\mathrm{Sk}_{\{x\}}\Big[\big(\exists y/\{x\}\big) x = y\Big] \quad &\text{is} \quad x = c, \\
\mathrm{Sk}\Big[\forall x \big(\exists y/\{x\}\big) x = y\Big] \quad &\text{is} \quad \forall x (x = c).
\end{aligned}$$

Skolem semantics

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Skolem semantics

Example

$$\exists x \big(\forall y/\{x\}\big) x \neq y$$

$$\begin{aligned} \mathrm{Sk}_{\{x,y\}}(x \neq y) &\quad \text{is} \quad x \neq y, \\ \mathrm{Sk}_{\{x\}}\Big[\big(\forall y/\{x\}\big) x \neq y\Big] &\quad \text{is} \quad \forall y (x \neq y), \\ \mathrm{Sk}\Big[\exists x \big(\forall y/\{x\}\big) x \neq y\Big] &\quad \text{is} \quad \forall y (c \neq y). \end{aligned}$$

Skolem semantics

Example

$$\exists x \big(\forall y / \{x\}\big) x \neq y$$

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Skolem semantics

Example

$$\forall x \exists z (\exists y/\{x\}) x = y,$$

$$\begin{aligned}
\mathrm{Sk}_{\{x,y,z\}}(x = y) &\quad \text{is} \quad x = y, \\
\mathrm{Sk}_{\{x,z\}}\left[(\exists y/\{x\}) x = y\right] &\quad \text{is} \quad x = g(z), \\
\mathrm{Sk}_{\{x\}}\left[\exists z (\exists y/\{x\}) x = y\right] &\quad \text{is} \quad x = g(f(x)), \\
\mathrm{Sk}\left[\forall x \exists z (\exists y/\{x\}) x = y\right] &\quad \text{is} \quad \forall x \left[x = g(f(x))\right].
\end{aligned}$$

Skolem semantics

Example

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\mathrm{Sk}_{\{x\}}\Big[\exists z (\exists y/\{x\}) x = y\Big] \quad & \text{is} \quad x = g\big(f(x)\big), \\
\mathrm{Sk}\Big[\forall x \exists z (\exists y/\{x\}) x = y\Big] \quad & \text{is} \quad \forall x \Big[x = g\big(f(x)\big)\Big].
\end{aligned}$$

Skolem semantics

Example

$$\forall x \exists y (\exists z/\{x,y\}) R(x,y,z),$$

$$\begin{aligned}
\mathrm{Sk}_{\{x,y,z\}}\big(R(x,y,z)\big) &\quad \text{is} \quad R(x,y,z), \\
\mathrm{Sk}_{\{x,y\}}\big[(\exists z/\{x,y\}) R(x,y,z)\big] &\quad \text{is} \quad R(x,y,c), \\
\mathrm{Sk}_{\{x\}}\big[\exists y (\exists z/\{x,y\}) R(x,y,z)\big] &\quad \text{is} \quad R(x,f(x),c), \\
\mathrm{Sk}\big[\forall x \exists y (\exists z/\{x,y\}) R(x,y,z)\big] &\quad \text{is} \quad \forall x\, R(x,f(x),c).
\end{aligned}$$

Skolem semantics

Example

$$\forall x \exists y (\exists z/\{x,y\}) R(x,y,z),$$

$$\begin{aligned}
\mathrm{Sk}_{\{x,y,z\}}(R(x,y,z)) &\quad \text{is} \quad R(x,y,z), \\
\mathrm{Sk}_{\{x,y\}}\Big[(\exists z/\{x,y\}) R(x,y,z)\Big] &\quad \text{is} \quad R(x,y,c), \\
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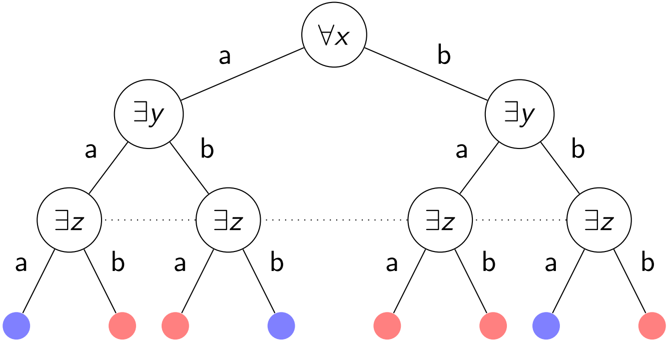
Skolem Semantics

$$\forall x\, R(x, f(x), c)$$



Infinity

Definition

An structure is *(Dedekind) infinite* if there exists a non-surjective injection from the universe to itself.

Infinity

Example

Let φ_∞ be the sentence,

$$\exists w \forall x (\exists y/\{w\})(\exists z/\{w,x\})[z = x \wedge y \neq w]$$

The Skolemization of φ_∞ is obtained in stages:

$$\begin{aligned} z = x \wedge y \neq w, \\ g(y) = x \wedge y \neq w, \\ g(f(x)) = x \wedge f(x) \neq w, \\ \forall x \Big[g(f(x)) = x \wedge f(x) \neq w \Big], \\ \forall x \Big[g(f(x)) = x \wedge f(x) \neq c \Big]. \end{aligned}$$

Infinity

Example

Let φ_∞ be the sentence,

$$\exists w \forall x (\exists y/\{w\})(\exists z/\{w, x\})[z = x \wedge y \neq w]$$

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Even

Definition

- An *involution* is a function that satisfies $f\big(f(x)\big) = x$ for all x in its domain.
- A finite structure has an even number of elements if and only if there is a way of pairing the elements without leaving any element out, i.e., if there exists an involution without a fixed point.

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Even

Example

Let φ_{even} be the IF sentence

$$\begin{aligned}\forall x \forall y (\exists u/\{y\})(\exists v/\{x,u\}) \Big[&(x = y \to u = v) \\ &\wedge (u = y \to v = x) \\ &\wedge u \neq x \Big].\end{aligned}$$

The Skolemization of φ_{even} is

$$\begin{aligned}\forall x \forall y \Big[&(x = y \to f(x) = g(y)) \\ &\wedge (f(x) = y \to g(y) = x) \\ &\wedge f(x) \neq x \Big].\end{aligned}$$

Even

Example

Let φ_{even} be the IF sentence

$$\begin{aligned}\forall x \forall y (\exists u/\{y\}) (\exists v/\{x, u\}) \Big[(x = y \rightarrow u = v) \\ \wedge (u = y \rightarrow v = x) \\ \wedge u \neq x\Big].\end{aligned}$$

The Skolemization of φ_{even} is

$$\begin{aligned}\forall x \forall y \Big[(x = y \rightarrow f(x) = g(y)) \\ \wedge (f(x) = y \rightarrow g(y) = x) \\ \wedge f(x) \neq x\Big].\end{aligned}$$

Even

Example (cont.)

The Skolemization of φ_{even},

$$\forall x \forall y \Big[\big(x = y \to f(x) = g(y)\big) \land \big(f(x) = y \to g(y) = x\big) \land f(x) \neq x \Big],$$

can be simplified to

$$\forall x \Big[f\big(f(x)\big) = x \land f(x) \neq x \Big].$$

Even

Example (cont.)

The Skolemization of φ_{even},

$$\begin{aligned}\forall x \forall y \Big[\big(x = y \rightarrow f(x) = g(y)\big) \\ \wedge \big(f(x) = y \rightarrow g(y) = x\big) \\ \wedge f(x) \neq x \Big],\end{aligned}$$

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Syntax

Definition

Given a fixed vocabulary and a set of variables, a language of *dependence logic* is generated by the following grammar:

$$\alpha \mid =(t_1, \ldots, t_n) \mid \neg\phi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\varphi \mid \forall x\varphi$$

Syntax

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Given a fixed vocabulary and a set of variables, a language of *dependence logic* is generated by the following grammar:

$$\alpha \mid {=}(t_1, \ldots, t_n) \mid \neg\phi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x \varphi \mid \forall x \varphi$$

Semantics

Definition

$$\mathbf{M}, X \models^{+} {=}(t_1, \dots, t_n) \quad \text{iff} \quad \text{any two assignments } s, s' \in X \text{ whose evaluations of the tuple } (t_1, \dots, t_{n-1}) \text{ coincide assign the same value to } t_n.$$

$$\mathbf{M}, X \models^{-} {=}(t_1, \dots, t_n) \quad \text{iff} \quad X \text{ is the empty team.}$$

Matching Pennies

Definition

In particular, the dependence atom $=\!(t)$ asserts

$$\mathbf{M}, X \models^{+} =\!(t) \quad \text{iff} \quad \text{every } s \in X \text{ assigns } t \text{ the same value.}$$

Matching Pennies

Example

$$\forall x \exists y \left[{=}(y) \wedge x = y \right]$$

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