Tableau metatheory for propositional logic

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In the presentation we explore the relationships between widely understood propositional logic and the tableau proofs. We set out a methodology for constructing adequate tableau systems for different sorts of propositional logic that can be determined with generalized relational semantics. The similar strategy, but relative to the context of syllogistic logic was implemented in the paper [8] where the tableau metatheory was developed for syllogistic languages.

The initial and basic idea that motivates our approach is the following observation: the syntax and semantics of a given logic always determine a minimal syntax and structure of a tableau system for the logic along with other properties. So, it seems reasonable to propose some metatheory for a class of logics that are very similar from a syntactical and semantic viewpoint. One can naturally ask: how can we benefit from such a theory? We show how the metatheory we propose makes the process of defining adequate tableau systems much easier. Since we define very general notions that cover all tableau issues concerning a very wide class of propositonal logics and demonstrate crucial relationships between them, we can apply them automatically in particular cases, concentrating only on remaining details, specific to the particular propositional logic under examination. The ideas behind the presented approach were outlined relative to certain contexts in [4], [5], [7], [6]. They have been developed and improved here to cope with propositional logic. Additional, and probably more important advantages of this metatheory that open future interesting research areas in tableau-proofs-approach are listed at the end.

At the beginning let us notice that in the theory of tableaux, we can distinguish three kinds of approaches to their construction.

First, tableau proofs are either with a signed or unsigned language. *Signed* means that in a language of tableau proofs, additional symbols to denote logical values are used, while *unsigned* means that in the tableau proofs we do not use any such symbols for logical values. This is a traditional division. It is worth mentioning that the first appearance of the tableau method in Beth [1] used signed tableaux (see also [11]).

Second, tableaux either can be built with languages that contain labels that denote possible worlds (points of relativization), or can be built with languages without labels (see [2], [3]).

Third, a way of construction of tableaux (and branches) can be divided into *nodes based on formulae* or *nodes based on sets of formulae*. While the former seems to be usually of didactic form (this approach is for example extensively outlined in [13]), the latter is more paradigmatic (see for example [2] and [3]) and has a strong connection to sequent calculi.

In our paper we set out a generalization of all these aspects of tableau theory for propositional logic that can be tweaked to any of the more specific forms mentioned above. We obtain the required generality by using *generalized labels* (but we call them *labels*). The generalized labels can obviously code points of relativization, but also other important semantic (and not only semantic) aspects, such as logical values, an object-property of belonging/non-belonging to a denotation of a given term or possibly other things. Other possible uses of generalized labels can be: (a) tracking the origin of decomposed formulae (see for example case for relevance tableaux [12] or paraconsistent tableaux [9], [10]), (b) quasi-negation (-)/quasi-assertion (+) in many valued tableaux or FDE tableaux, including Routley Star (see for example [13]). Moreover, since we are developing a metatheory, we should be open to new roles for labels, which may only emerge when we use the metatheory for particular cases of new propositional logics.

Last but not least, our approach is of the nodes-based-on-sets-of-formulae kind but different to others of this kind in at least two ways. First, decomposed expressions are collected rather than deleted, so at any stage of a tableau branch, the full information on the proof is still available. So additional constraints on tableau rules can be imposed directly on the inputs of rules (since in our approach tableau rules are sets of n-tuples of sets such that the input is always a proper subset of the output). Second, in fact we do not use direct tree structures with nodes, since branches are strictly monotonic sequences of input/output sets, while tableaux are sets of such branches (selected with some additional conditions). However, the remaining approaches (with nodes based on formulae as well as sets of formulae) can be defined using our apparatus, since our approach is more abstract.

In the presentation we determine a language that includes what is common to any propositional logic, whilst not excluding richer grammatical constructions, since we want our theory to cover all possible propositional languages.

Then we introduce general semantic structures for propositional logics, as well as notions of satisfiability which together form models. It turns out that models semantically determine particular propositional logics, which is illustrated by examples. In keeping with the spirit of generality, we propose a syntax for our tableau language to describe properties of general semantic structures in tableau proofs. Finally, we introduce a notion of a set of satisfied expressions.

The next part is completely devoted to problems of what is generally a tableau rule, a branch, a tableau etc. A multistage, set-theoretical construction of these notions is proposed. As a novelty in the field of propositional logic a branch consequence relation as the tableau counterpart of a semantic consequence relation for a given logic is also introduced.

All important relationships between these notions are proved to present general connections between the tableau notions and general semantics. This establishes sufficient conditions for a complete and sound tableau system that are in this section formulated.

In the last part some perspectives for the further development of the tableau metatheory are presented.

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