Interpolation and proof systems

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There are various reasons for developing proof systems for logics. They may, for example, be used to prove that a logic is consistent or decidable, or provide a means to uncover certain structural properties of a logic, such as interpolation.

Interpolation is considered by many to be a "good" logical property because it indicates a certain well-behavedness of the logic, vaguely reminiscent to analycity: if an implication $\varphi \to \psi$ holds in the logic, then there is a formula χ in the common language of φ and ψ that *interpolates* the given implication, that is, such that $\varphi \to \chi$ and $\chi \to \psi$ hold. What the common language is depends on the logic one considers. In propositional logics it typically means that all atoms in χ occur in φ as well as in ψ.

As expected, many well-known logics satisfy interpolation, such as classical propositional and predicate logic, which was shown by William Craig in 1957. More than three decades later it turned out that some of the standard logics with interpolation also satisfy the stronger property of *uniform interpolation*, where the interpolant only depends on the premiss or the conclusion of the given implication.

Whereas in the presence of a decent analytic proof system, proofs of interpolation are often relatively straightforward, proofs of uniform interpolation are in general quite complex. In this talk I will describe a method to extract uniform interpolants from sequent calculi and prove, using this method, that logics without uniform interpolation lack certain calculi. Thus having uniform interpolation becomes a property of proof systems rather than of logics. The method applies to many propositional logics, including modal and intermediate logics, and thereby provides a way to prove that several of such logics do not have proof systems of a certain form.