$\{0,1\}$ AND $[0,1]$: FROM CLASSICAL LOGIC TO FUZZY QUANTUM LOGIC

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The starting point of the unsharp approach to quantum mechanics (QM) ([2]) is deeply connected with a general problem that naturally arises in the framework of Hilbert space quantum theory. Let us consider an event-state system $(\Pi(\mathcal{H}), \mathcal{S}(\mathcal{H}))$, where $\Pi(\mathcal{H})$ is the set of **projections**, while $\mathcal{S}(\mathcal{H})$ is the set of all **density operators** of the Hilbert space $\mathcal{H}$ (associated to the physical system under investigation). Do the sets $\Pi(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ correspond to an *optimal* possible choice of adequate mathematical representatives for the intuitive notions of *event* and of *state*, respectively? Once $\Pi(\mathcal{H})$ is fixed, Gleason's Theorem guarantees that $\mathcal{S}(\mathcal{H})$ corresponds to an *optimal* notion of state: for, any probability measure defined on $\Pi(\mathcal{H})$ is determined by a density operator of $\mathcal{H}$ (provided the dimension of $\mathcal{H}$ is greater than or equal to 3). On the contrary, $\Pi(\mathcal{H})$ does not represent the largest set of operators assigned a probability-value since there are bounded linear operators E of $\mathcal{H}$ that are not projections and that satisfy the *Born's rule*: for any density operator ρ, $\mathrm{Tr}(\rho E) \in [0,1]$. In the unsharp approach to QM, the notion of *quantum event* is liberalized and the set $\Pi(\mathcal{H})$ is replaced by the set of all *effects* of $\mathcal{H}$ (denoted by $\mathcal{E}(\mathcal{H})$), where an effect of $\mathcal{H}$ is a bounded linear operator E that satisfies the following condition, for any density operator ρ : $\mathrm{Tr}(\rho E) \in [0,1]$. Clearly, $\mathcal{E}(\mathcal{H})$ properly includes $\Pi(\mathcal{H})$.

The set $\mathcal{E}(\mathcal{H})$ can be naturally structured ([1],[2]) as a *Brouwer-Zadeh poset* (BZ-poset) $\langle \mathcal{E}(\mathcal{H}), \leq, ', {}^{\sim}, \mathbb{O}, \mathbb{I} \rangle$, where

- (i) $E \leq F$ iff for any density operator $\rho \in \mathcal{S}(\mathcal{H}) : \mathrm{Tr}(\rho E) \leq \mathrm{Tr}(\rho F)$;
- (ii) $E' = \mathbb{I} - E$ (where $-$ is the standard operator difference);
- (iii) $E^{\sim} = P_{Ker(E)}$, where $P_{Ker(E)}$ is the projection associated to the kernel of E;
- (iv) $\mathbb{O}$ and $\mathbb{I}$ are the null and the identity projections, respectively.

The BZ-poset $\mathcal{E}(\mathcal{H})$ turns out to be properly fuzzy since the noncontradiction principle is violated ($E \wedge E' \neq \mathbb{O}$). Further, the BZ-poset $\mathcal{E}(\mathcal{H})$ fails to be a lattice ([2]). In a quite neglected paper, however, Olson ([4]) proved that $\mathcal{E}(\mathcal{H})$ can be equipped with a natural partial order $\leq_s$ (called *spectral order*) in such a way that $\langle \mathcal{E}(\mathcal{H}), \leq_s \rangle$ turns out to be a *complete lattice*. In this talk, we will present the algebraic properties of the structure $\langle \mathcal{E}(\mathcal{H}), \leq_s, ', {}^{\sim}, \mathbb{O}, \mathbb{I} \rangle$ and we will introduce a new class of BZ-lattices (called $BZ^* - lattices$) that represents a quite faithful abstraction of the concrete model based on $\mathcal{E}(\mathcal{H})$ (see also [3]). Interestingly enough, in the framework of BZ*-lattices different abstract notions of "unsharpness" collapse into the one and the same concept, similarly to what happens in the concrete BZ*-lattices of all effects ([5, 6]). We will finally present the structure theory of PBZ*-lattices and we provide an initial description of the lattice of PBZ*-varieties.

References

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